Alternative Trigonometric Solutions for Rietz-type Slide Rules

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Introduction

This paper investigates the possibility of using the trigonometric scales in a manner not usually described in manuals or books but first, some background material is needed to describe the basis for the alternatives.

Reciprocals -- A reciprocal is a quotient with the dividend equal to 1 or a ratio with numerator as 1 and the denominator as the other factor. On a slide rule if the number is set on the C scale over the D index, the quotient is found on the D scale under the opposite C index. I call this operation “index exchange”. This property was described in [4], [5], and [6] but the authors do not exploit it in solutions. Later books manuals do not appear to describe this property nor use it. This property can play a convenient role in some applications, as we will see later.

Recently, Jeremy Kelly in the International Slide Rule Group (ISRG) message 40942 asked how to do exponentiation calculations on a Darmstadt rule. Such rules are a solid body simplex with log-log scales on the back of the slide. These rules have an opening or gap in the gutter or valley at each end usually filled by a clear window with a hairline aligned with the index marks on the face of the rule similar to Rietz-type rules. Normally the Darmstadt rule manuals suggest removing the slide, flipping it over and reinstalling it. Computations are then done more or less directly using the D scale. Virtually all manuals and books recommend using a Darmstadt rule in this manner.

After seeing this solution, I wondered if Rietz-type slide rules could handle trigonometric functions is a similar manner. This would be different than what is commonly used in virtually all manuals and books which recommend flipping the slide over to do trigonometric computations. A very good description of using the trigonometric scales in this manner can be seen in [1]. As I have not spent any time learning the details of trigonometric solutions on Rietz-type rules, my curiosity was whetted. So, began an interesting investigation. See [8] for a description of the Rietz System.

In the following descriptions, I refer to a later model where there are three trigonometric scales on the back of the rule. The methods described can be easily adapted to earlier rules, which used the double cycle sine S scale tied to the A-B scales. The trig scales transfer settings from the back of the slide to the front by index marks or window-based hairlines called a back index [4] aligned with the D index on the front.

Rietz Trigonometric Solutions From Back of Slide

I have divided the results of my investigation into four parts. The first covers the common trigonometric functions sine, cosine and tangent. The second covers the less common ones cosecant, secant, and cotangent. The third covers the arc-functions while the last covers the complex number operations as an application example.

Sine, Cosine, and Tangent Functions

When I started, I tried using conventional methods and quickly encountered some mental gymnastics. So, I resorted to my usual approach when I begin to tackle problems on a slide rule with unfamiliar scale arrangements, I start with developing proportional equations to see what scale settings of the new rule match the ratios needed and what pre- and post-operations may be needed to complete the computation. If you are not familiar with using proportional methods on function scales, see [3].
I show these for the basic operations sine, cosine, and tangent in Table [1]. The first column of these operations defines the expression to evaluate. The second column shows the derivation of the proportion equation. The third column shows the symbolic cursor and slide settings. Finally, the last column shows an example problem solution. Every solution starts with a slide setting done on the back of the rule, e. g., $\Rightarrow S(A)$ set angle A on the S scale to the back index. The rule needs to be flipped over to continue the operations as given. Note that some of these solutions involve index exchange operations. These are shown by a **>** after the setting. This may be done because the proportion answer is given as the reciprocal of the true answer on the D scale. By using the index exchange operation, the true answer can read on the opposite scale index -- see sinA/x in Table [1] and $r = a\sin A/\sin B$ [2] for examples.

Table 1. Basic Trig Functions[1]

Expression	Solution Steps			
	Proportion solution	Symbolic Solution	Numeric Solution A=25°, x=3	

$x\sin A$	$z = x\sin A$ $\frac{\sin A}{1} = \frac{z}{x}$ $\frac{C(\sin A)}{D(10)} = \frac{C(x\sin A)}{D(x)}$	$\Rightarrow S(A)$ $\rightarrow D(x)$ $\bullet C(x\sin A)$	$\Rightarrow S(25)$ $\rightarrow D(3)$ $\bullet C(1.27)$	
$\frac{x}{\sin A}$	$z = \frac{x}{\sin A}$ $\frac{\sin A}{1} = \frac{x}{z}$ $\frac{C(\sin A)}{D(10)} = \frac{C(x)}{D(x/\sin A)}$	$\Rightarrow S(A)$ $\rightarrow C(x)$ $\bullet D(x/\sin A)$	$\Rightarrow S(25)$ $\rightarrow C(3)$ $\bullet D(7.10)$	
$\frac{\sin A}{x}$	$z = \frac{\sin A}{x}$ $\frac{\sin A}{1} = \frac{x}{1/z}$ $\frac{C(\sin A)}{D(10)} = \frac{C(x)}{D(x/\sin A)}$	$\Rightarrow S(A)$ $\rightarrow C(x)$ $\Rightarrow C(10)$ $\rightarrow D(1)$ $\bullet C(\sin A/x)$	$\Rightarrow S(25)$ $\rightarrow C(3)$ $\Rightarrow C(10)$ $\rightarrow D(1)$ > $\bullet C(.141)$	
$r = a\sin A/\sin B$	$r = \frac{a}{\sin B}\sin A$ $\frac{r}{\left(a\frac{1}{\sin B}\right)} = \frac{\sin A}{1}$ $\frac{C(r)}{D(a\frac{1}{\sin B})} = \frac{C(\sin A)}{D(10)}$	$\Rightarrow S(B)$ $\rightarrow C(a)$ $\Rightarrow S(A)$ $\bullet C(r)$	$\Rightarrow S(45)$ $\rightarrow C(3)$ $\Rightarrow S(30)$ $\bullet C(2.12)$	

Tangent operations differ as the function value is found under the C index rather than the D index after the angle is set on the T or T′ scale

Tabel 2. Tangent Operations for Angles >45°

Expression	Tangent A>45°			
	Proportion Solution	Symbolic Solution	Numeric Solution	

			A=65°, x=3	
$x\tan A$	$z = x\tan A$ $\frac{\tan A}{1} = \frac{z}{x}, \frac{1}{\tan A} = \frac{x}{z}$ $\frac{C(1)}{D(\tan A)} = \frac{C(x)}{D(x\tan A)}$	$\Rightarrow T'(A)$ $\rightarrow C(x)$ $\bullet D(x\tan A)$	$\Rightarrow T'(65)$ $\rightarrow C(3)$ $\bullet D(6.43)$	
$\frac{x}{\tan}$	$z = \frac{x}{\tan A}, \tan A = \frac{x}{z}$ $\frac{1}{\tan A} = \frac{z}{x}$ $\frac{C(1)}{D(\tan A)} = \frac{C(x/\tan A)}{D(3)}$	$\Rightarrow T'(A)$ $\rightarrow D(x)$ $\bullet C(x/\tan A)$	$\Rightarrow T'(65)$ $\rightarrow D(3)$ $\bullet C(1.40)$	
$\frac{\tan A}{x}$	$z = \frac{\tan A}{x}$ $\frac{1}{\tan A} = \frac{1}{zx} = \frac{1/x}{z}$ $\frac{C(1)}{D(\tan A)} = \frac{CI(x)}{D(\tan A/x)}$	$\Rightarrow T'(A)$ $\rightarrow CI(x)$ $\bullet D(\tan A/x)$	$\Rightarrow T'(65)$ $\rightarrow CI(3)$ $\bullet D(0.715)$	

Cosecant, Secant and Cotangent Functions – These are not often used but the description of them can be found in JOS Plus article. They tend to be much shorter as these functions are the reciprocal of sine, cosine, and tangent and are found automatically under the C index using the index exchange method.

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Arc Angle Functions -- The set ups for arc angle computations are quite straightforward as the argument needs to be set on the C scale over the D index with the exception of the arctangent for arguments greater than 1.0 where the C(1) is set over the argument on the D scale. The flip the rule over and read the angle on the appropriate S, S', T or T' scales under the back index. For arguments between 0.01 and 0.1, use the ST scale.

Complex Number Computation -- I have for some time wondered how Rietz-type rules handled complex number computations as having the trigonometric scales on the back of the rule appeared to be troublesome. Several months ago, I saw a photo of Steinmetz, who developed the use of complex number for alternating current circuit solutions, in his small rowboat on one of those long lakes in central New York. On the seat next to him was a small slide rule most likely a K&E Polyphase. How did he use this rule?

I set up the usual equations in proportional form and translated them into the Rietz scale forms shown in Table [3]. Because the trig function values are found on the C scale over the D index, one of the proportion ratios is sinA/1 or tanA/1 which is natural for the Rietz rule. The results, to some extent, surprised me, as they are quite efficient. So this simple appearing slide rule could be quite useful for some serious computations. By keeping the trigonometric scales on the back and not flipping it over left the slide in place with the C, CI and other scales ready to complete an extended calculation the user may have needed. Flipping the rule over to do the conversion and then flipping it back would slow the computation somewhat.

Table 3. Complex Number Conversion

Rectanglular Form	Proportion Angle Equation	Hypotenuse Equation	Polar Form
$x + jy$	$\tan A = \frac{y}{x}$ $\frac{\tan A}{1} = \frac{C(y)}{D(x)}$	$\frac{y}{r} = \sin A$ $\frac{y}{r} = \frac{\sin A}{1}$	$r/\underline{A}$
$4 + j3$ x>y A<45°	$\rightarrow D(x = 4)$ larger x,y $\Rightarrow C(y = 3)$ smaller x,y $\bullet T(A = 36.9)$	$\Rightarrow S(A = 36.9)$ $\rightarrow C(y = 3)$ smaller x, y $\bullet D(r = 5)$	$5/\underline{36.9}$
$3 + j4$ x<y A>45°	$\rightarrow D(y = 4)$ larger x,y $\Rightarrow C(x = 3)$ smaller x,y $\bullet T'(A = 53.1)$	$\Rightarrow S'(A = 53.1)$ $\rightarrow C(x = 3)$ smaller x, y $\bullet D(r = 5)$	$5/\underline{53.1}$

Summary -- Using the trig scales from the backside of the slide on Rietz-type slide rules is possible and convenient in most cases. I found that the Rietz-type rules could be used for trigonometric computations quite efficiently without removing the slide, flipping it over and reinstalling it. In general, the solutions are comparable to those found on rules with the trig scales on the front of the rule. In some cases, they may be a little less efficient due to the starting point over the D index on the C scale. This is due to having to use the reciprocal of the trigonometric function values. These solutions have the advantage of leaving the slide in its normal position so that additional computations can be done without the time loss due to flipping the slide over. Computation of complex numbers is just as efficient as other rules, which surprised me.

There are, however, a lot of triangle computations the use the law of sines that can't be done with the trig scales left to the back. See reference [1] as a good source. Many computations can be done that may involve, for example, two sine functions. A good example is: $r = a\sin A / \sin B$. Conventional slide rules with the trig scales on the face of the slide can't handle this easily as two settings of the S scale are called for simultaneously. The Reitz configuration can, however, manage this because the back index provides one of the S scale settings where the conventional rule has only the cursor hairline. As an exercise, see if you can derive the proportion equation for a three step solution.

Acknowledgments

I want to thank Clark McCoy and John Mosand for notes on the history of Rietz and Polyphase slide rules as well as [8]. In addition, thanks to Peter Johnson and João Gabbardo for pointing out references [5] and [6] that described the index exchange methods for a variety of reciprocal slide rule operations. Reference [5] is particularly rich is examples of using reciprocals – a topic unfortunately dropped by later books and manuals.
I used three different Rietz-type rules for developing the solutions: A Russian LSLO-25-10P (made in 1987!), a Post 1447, and a UTO 601. Each of these rules has a different back index configuration.

Slide Rule Operations Notation– I use a shorthand notation for describing slide rule operations. It starts by recognizing that there are three basic operations used in any solution – 1)moving the hairline, 2)moving the slide and 3)reading the result on some scale. So,
1) Move the hairline to 45 on the C scale: ->C(45).
2) Move the slide so 673 is under the hairline on the D scale: =>D(673).
3) Read the result on the T scale: •T(35.9).
The index marks can read as either 1 or 10 (such as C(1) or C(10)) as needed).
The scale S reads from left to right for sine values. Cosine values area read on the S scale from right to left by denoting S′. T′ is used for angles greater than 45°.

Notes:

1. In this table, all of the definitions are shown for sin functions but the descriptions can be used for cos and tan (where A is less than 45°) as needed by replacing sin by cos or tan. Scale references change for cos and become S′. This is done to save space in this article. Complete definitions for all combinations can be found on JOS Plus.
2. This is taken from Ronald Manley's web site http://www.sliderules.clara.net/a-to-z/trig-02.htm.
   I added the proportional equation to show the derivation of the settings.

Note that is an example where Rietz-type rules provide the capability of having two trig function values set on the rule: one on the C scale projected from the back and one on the D scale from the index exchange of the C scale value – the feature is not readily available on most other rules.

References:

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